Extended Kalman Filter

with slides adapted from http://www.probabilistic-robotics.com

Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + n^2)$$

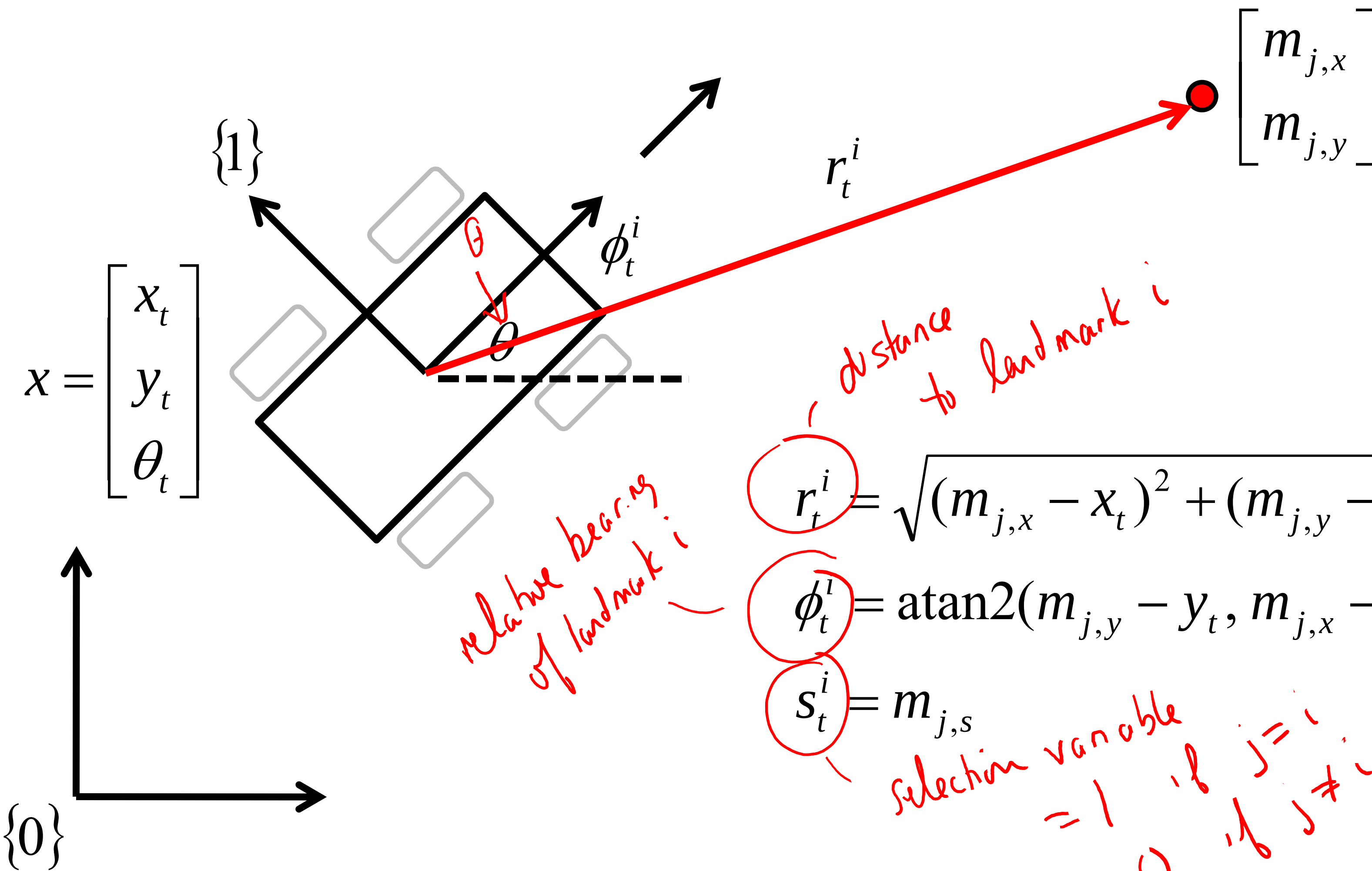
optimal matrix multiplication.

- Optimal for linear Gaussian systems!

- Most robotics systems are nonlinear!

Landmark Measurements

- distance, bearing, and correspondence



distance to landmark i

$$r_t^i = \sqrt{(m_{j,x} - x_t)^2 + (m_{j,y} - y_t)^2}$$

relative bearing of landmark i

$$\phi_t^i = \operatorname{atan2}(m_{j,y} - y_t, m_{j,x} - x_t) - \theta_t$$

$$s_t^i = m_{j,s}$$

selection variable

=1 if j=i

=0 if j≠i

Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

$$\boxed{x_t = g(u_t, x_{t-1})}$$

non-linear in u_t and x_t

$$\boxed{z_t = h(x_t)}$$

non-linear in x_t

Nonlinear Dynamic Systems

- localization with landmarks

$$x_t = \underbrace{\begin{pmatrix} x - \frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t\,\Delta t) \\ y + \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t\,\Delta t) \\ \theta + \omega_t\,\Delta t \end{pmatrix}}_{g(u_t, x_{t-1})}$$

forward kinematics for differential drive

$$\underbrace{\begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix}}_{z_t^i} = \underbrace{\begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y_t, m_{j,x} - x_t) - \theta \\ m_{j,s} \end{pmatrix}}_{h(x_t, j, m)}$$

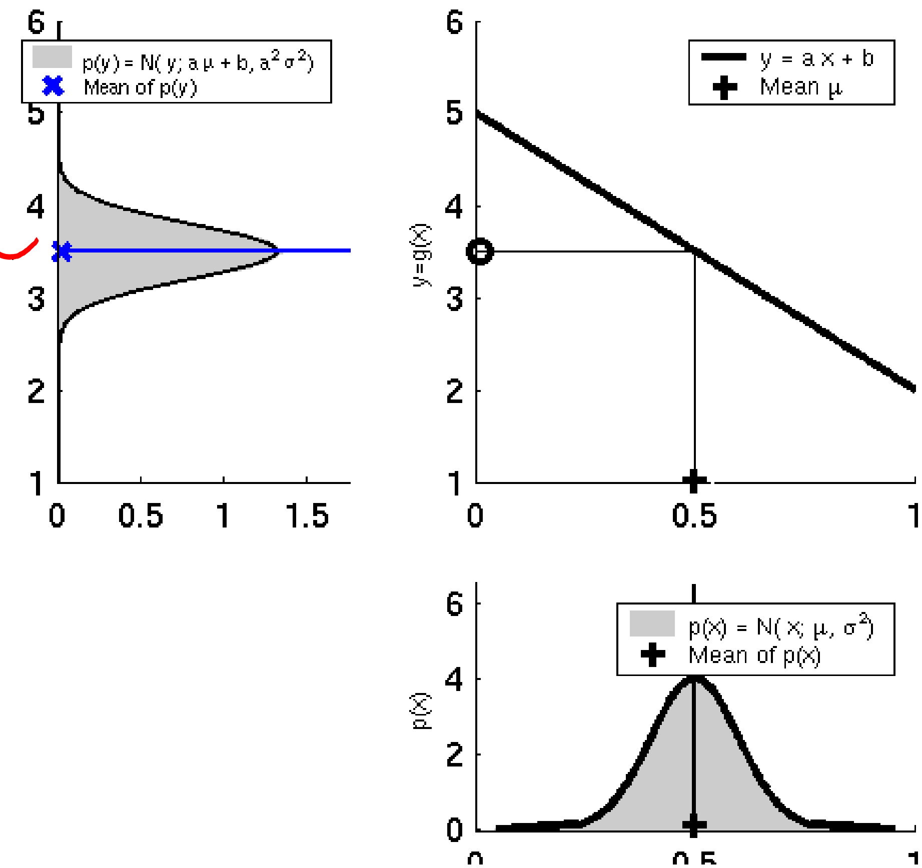
Linearity Assumption Revisited

predicted state

$\bar{\mu}_t$

$\bar{\Sigma}_t$

predicted covariance of state



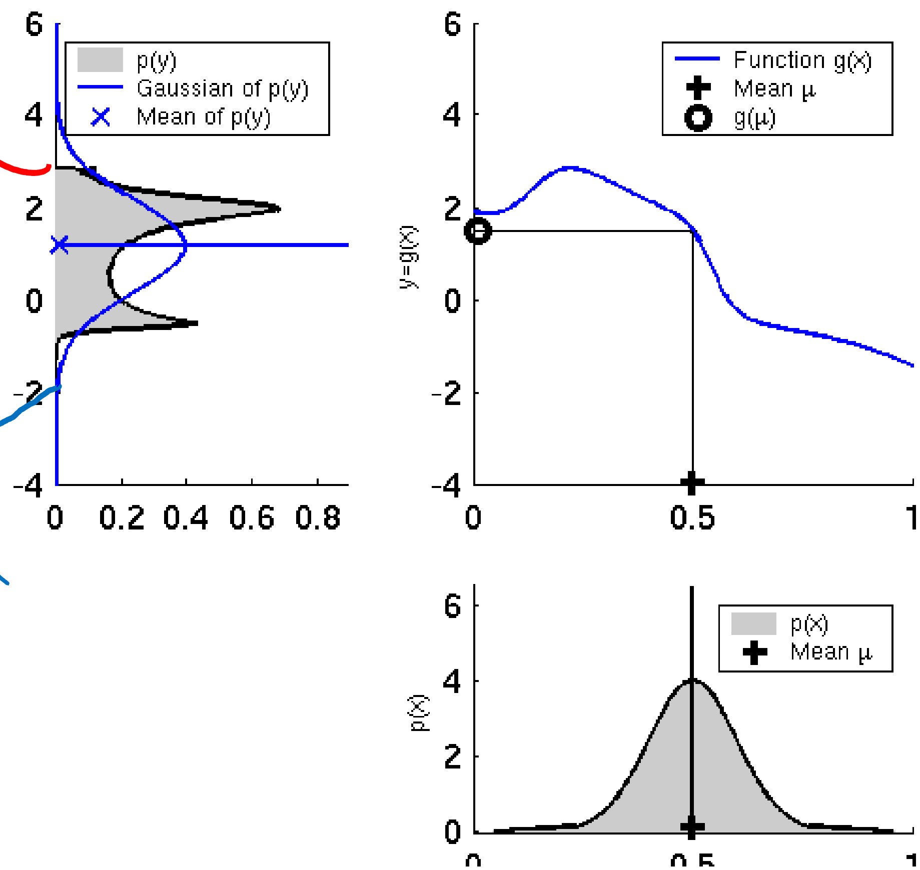
linear plant or observation model

μ_{t-1}

Σ_{t-1}

Non-linear Function

Gaussian distribution is no longer Gaussian



$\bar{\mu}_t$

$\bar{\Sigma}_t$

KF assumes distribution is Gaussian

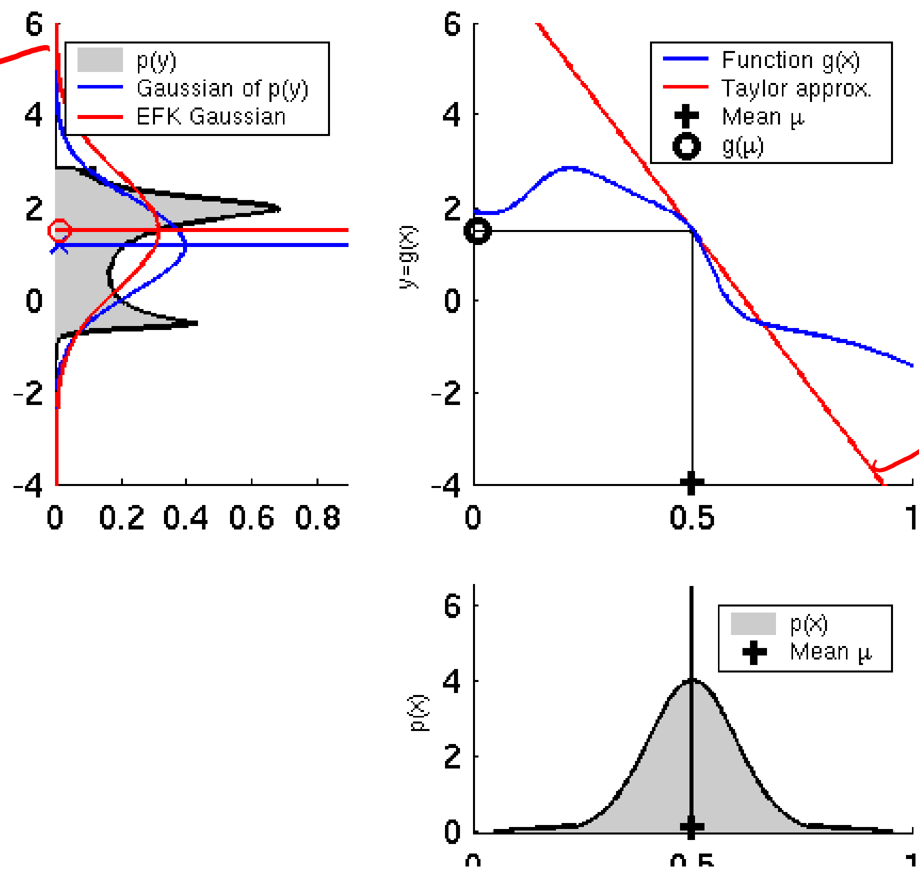
μ_{t-1}

Σ_{t-1}

EKF Linearization (1)

$\bar{\mu}_t, \bar{\Sigma}_t$

fair approximation of the distribution



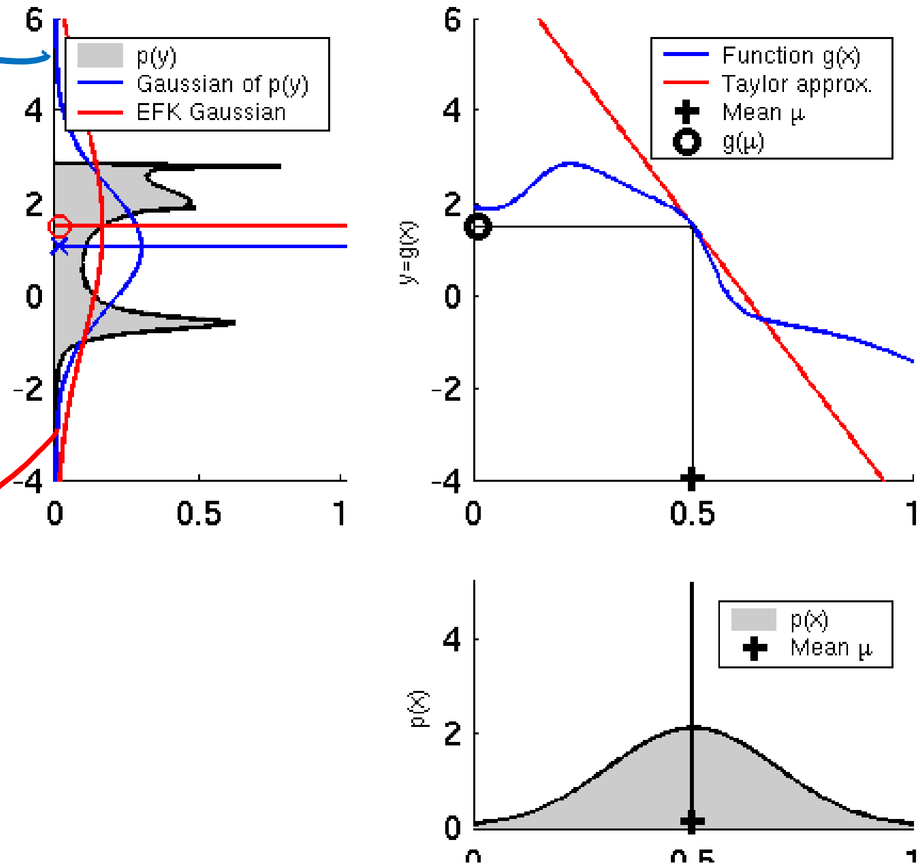
linear approximation of g(x)

μ_{t-1}

Σ_{t-1}

EKF Linearization (2)

$\bar{\mu}_t, \bar{\Sigma}_t$ computed by finding the mean and covariance of gray dist.



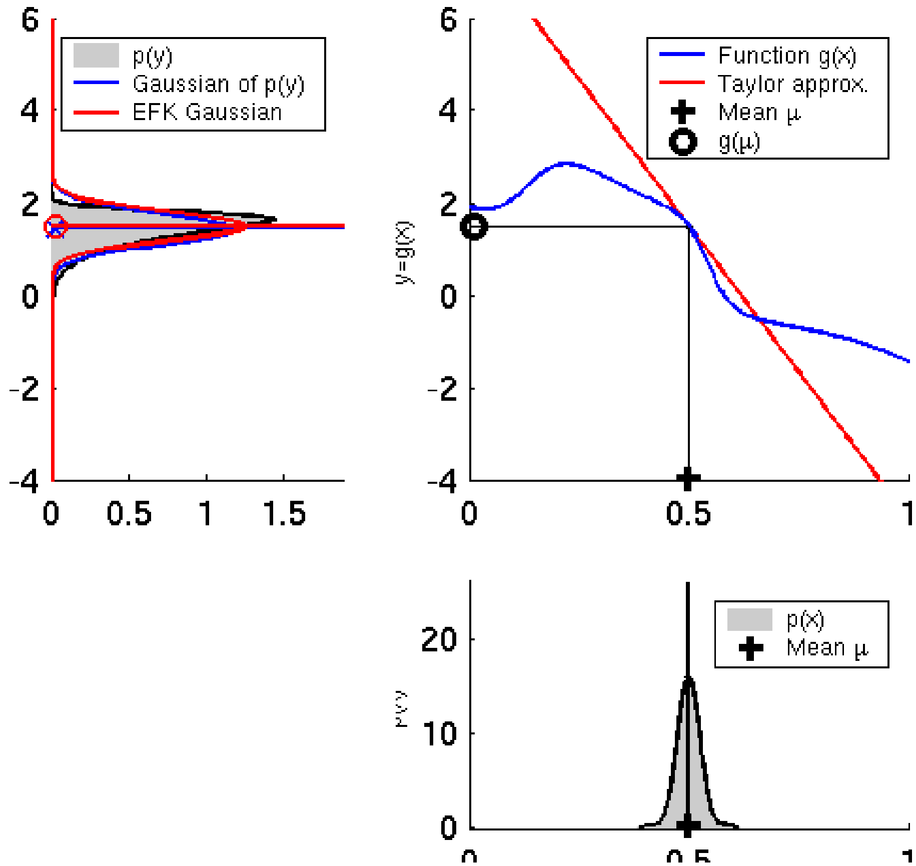
$\bar{\mu}_t, \bar{\Sigma}_t$ computed using the linear approx.

μ_{t-1}

Σ_{t-1}

where Σ_{t-1} is wide

EKF Linearization (3)



Taylor Series

- recall for f(x) infinitely differentiable around in a neighborhood a

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$
$$\approx f(a) + f'(a)(x-a)$$

- in the multidimensional case, we need the matrix of first partial derivatives (the Jacobian matrix)

EKF Linearization: First Order Taylor Series Expansion

- Prediction:

Jacobian of plant model

plant model

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

- Correction:

Jacobian of measurement model

measurement model

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

EKF Algorithm

1. **Extended_Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

Kalman filter

2. Prediction:

Jacobian

3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ $\longleftarrow$ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$ $\longleftarrow$ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

Jacobian

6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$ $\longleftarrow$ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ $\longleftarrow$ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ $\longleftarrow$ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

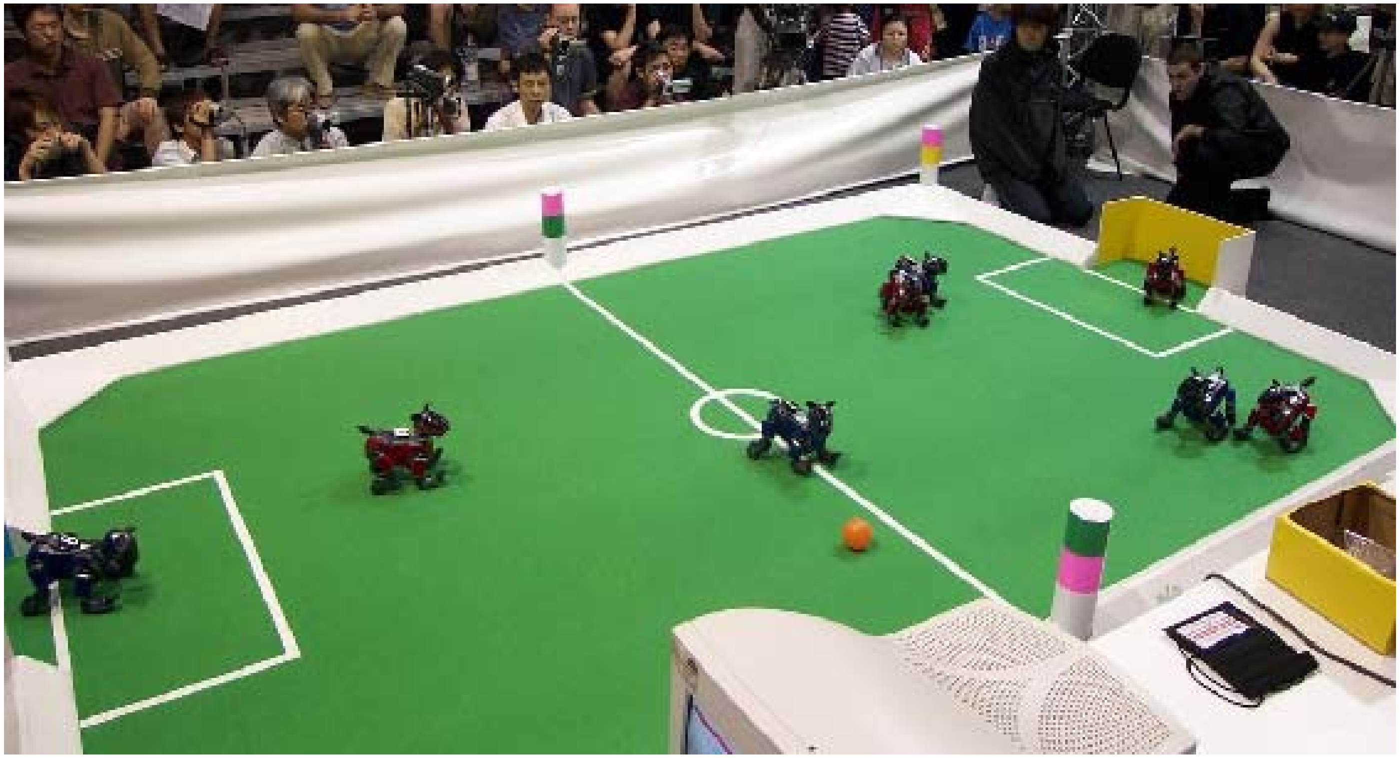
$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

Localization

"Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities." [Cox '91]

- **Given**
  - Map of the environment. i.e. landmarks
  - Sequence of sensor measurements.
- **Wanted**
  - Estimate of the robot's position.
- **Problem classes**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)

Landmark-based Localization



Revisit omnibot example

$$x_t = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_{t-1}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}_t}_{u_t} + \varepsilon_t$$

plant model is linear

$$z_t = \begin{bmatrix} L_x \\ L_y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}_t + \delta_t$$

$$H = \begin{bmatrix} \frac{\partial}{\partial x}(L_x - x + \delta_{x,t}) \\ \frac{\partial}{\partial y}(L_y - y + \delta_{y,t}) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

1. **EKF_localization** ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

$$X_t = \begin{bmatrix} x'(x_{t-1}, u_t) \\ y'(x_{t-1}, u_t) \\ \theta'(x_{t-1}, u_t) \end{bmatrix}$$

Prediction:

3. $$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$$ Jacobian of *g* w.r.t location

5. $$V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$ Jacobian of *g* w.r.t control

6. $$M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$ Motion noise

7. $$\bar{\mu}_t = g(u_t, \mu_{t-1})$$ Predicted mean

8. $$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$$ Predicted covariance

1. **EKF_localization** (μ_{t-1}, Σ_{t-1}, u_t, z_t, m):

Correction:

3. $$\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$$ Predicted measurement mean

(note: not using selection variable)

5. $$H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$$ Jacobian of *h* w.r.t location

6. $$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

7. $$S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$$ Pred. measurement covariance

8. $$K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$$ Kalman gain

9. $$\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$$ Updated mean

10. $$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$$ Updated covariance